Evolution of cosmological perturbations in the universe dominated by multiple scalar fields

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Abstract

By efforts of several reserchers, it is recently established that the dynamical behavior of the cosmological perturbation on superhorizon scales is well approximated in terms of that in the long wavelength limit, and that the latter can be constructed from the evolution of corresponding exactly homogeneous universe. Using these facts, we investigate the evolution of the cosmological perturbation on superhorizon scales in the universe dominated by oscillating multiple scalar fields which are generally interacting with each other, and the ratio of whose masses is incommensurable. Since the scalar fields oscillate rapidly around the local minimum of the potential, we use the action angle variables. We found that this problem can be formulated as the canonical perturbation theory in which the perturbed part appearing as the result of the expansion of the universe and the interaction of the scalar fields is bounded by the negative power of time. We show that by constructing the canonical transformations properly, the transformed hamiltonian becomes simple enough to be solved. As the result of the investigation using the long wavelength limit and the canonical perturbation theory, under the sufficiently general conditions, we prove that for the adibatic growing mode the Bardeen parameter stays constant and that for all the other modes the Bardeen parameter decays. From the viewpoint of the ergodic theory, it is discussed that as for the Bardeen parameter, the singularities appear probabilistically. This analysis serves the understanding of the evolution of the cosmological perturbations on superhorizon scales during reheating.

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§1 Introduction

The cosmological structures such as the galaxies and the clusters of galaxy, etc, are thought to be produced from the seed pertubations through the growth induced by the gravitational instability. In the inflationary paradigm, the quantum fluctuation of the scalar field driving the inflation called the inflaton plays the role of the seed perturbation. In the slow rolling phase, the seed perturbations produced in the Hubble horizon are streched into the superhorizon scales by exponential cosmic expansion, and stay out of horizon until the perturbation comes into horizon in the Friedman regime. We can calculate the amplitude of the seed pertubations when the potential of the inflaton is given, and know the evolutionary behavior of the cosmological perturbation in the horizon in the Friedmann universe when the composition of the cosmic matter is given. Therefore in order to obtain the knowledge of the inflationary universe from the statistical informations of the galaxy distribution, we need to know how the amplitudes of pertubations on superhorizon scales grow from the first horizon crossing in the slow rolling phase until the second horizon crossing in the Friedmann regime.

The evolution of the cosmological perturbation on the superhorizon scales are thought to be simple. Because the Bardeen parameter are conserved in a good accuracy, when the equation of state is regular, and the energy of the universe is dominated by a single component. [3] But between the inflationary regime and the Friedmann regime, reheating occurs. During reheating, since the plural scalar fields oscillate rapidly around the local minimum of the potential, and since radiation is generated by the energy transfer from the scalar field oscillation, the equation of state becomes singular periodically [7], and the entropy modes can appear [8]. Therefore the approximate conservation of the Bardeen parameter is not self-evident, and we need to solve the evolution of the cosmological perturbations without using the conservation of the Bardeen parameter. In general inflationary scenarios, there exisists plural scalar fields. [12] So in this paper, we consider the universe dominated by multiple scalar fields.

Recently the strong method of solving the evolution of cosmological perturbations on superhorizon scales is developed. The evolutionary behavior of the cosmological perturbations on superhorizon scales is described by that in the long wavelength limit in a good accuracy. [7] Further the long wavelength limit of the solution of the evolution equation of the cosmological perturbations is generated from the perturbation of solution of the evolution equation of the background homogeneous universe. [4], [5], [6]. By using the fact, we investigate the evolution of the cosmological perturbation on superhorizon scales in the universe dominated by the oscillating multiple scalar fields interacting with each other.

In order to use the above method, we are required to solve the exactly homogeneous system accurately. During reheating, the scalar fields oscillate rapidly around the local minimum of the potential. As a method of treating dynamical system with rapidly oscillating degrees of freedom, we use the action angle variables, the hamiltonian formulation and the canonical perturbation theory. [9] In the canonical perturbation theory, the hamiltonian is decomposed into an unperturbed part whose solution is already known and an perturbation part, and by constructing the change of dynamical variables by which pertubation part is decreased, the system is made simple enough to be solved. The dynamical degrees

of freedom are decomposed into the slowly varying amplitudes of oscillation called action variables which are involutive first integrals of the unperturbed part and the fast increasing phase of oscillation called angle variables. The angle variables are eliminated from the hamiltonian by the canonical transformation. By using the method of the long wavelength limit and the canonical perturbation theory, we investigate the evolutionay behavior of the Bardeen parameter. While we adopt the action angle variables as the dynamical variables in the viewpoint of treating the reheating, from the standpoint decomposing the adiabatic and the entropy perturbation, the different variables were proposed. [11] Also in other approach the multiple scalar fields system was investigated. [10]

This paper is organized as follows. In chapter 2, using the result of the paper [5], we explain the way to construct a solution of the perturbation equation in the long wavelength limit from a perturbation of a solution of the exactly homogeneous universe. In chapter 3, we write the system of the multiple scalar fields in the homogeneous universe in the hamiltonian form and fix the model which we treat concretely. In chapter 4, we apply the canonical perturbation theory to our system. The scalar fields oscillation is decomposed into the action angle variables, the former of which and the latter of which represent the slowly varying amplitudes and fast increasing phases of the scalar fields oscillation, respectively. By using the canonical transformations reccurently, we decrease the power of time variable which the interaction parts contain and eliminate fast oscillation from the evolution equation and extract the slow secular motion. We give the evaluation of the generating functions of the canonical transformations by which the transformed system becomes simple enough to be solved. In chapter 5, we investigate the evolution of the multiple scalar fields in the homogeneous universe. We prove the fact that the action variables are perpetually stable. In chapter 6, we investigate the evolution of the cosmological perturbations in the long wavelength limit paying attention to that of the Bardeen parameter. The upper bound of evolution of the perturbation variables and the uniqueness of the growing mode are derived. As a result, it is concluded that the Bardeen parameter stays constant in a good accuracy when the masses of the scalar fields are incommensurable. In this case, the Bardeen parameter becomes singular probabilistically. So in terms of the ergodic theory, we discuss the distribution of the singularity of the Bardeen parameter. The chapter 7 is devoted to the summary and the conclusion.

Throughout the paper the natural units $c = \hbar = 1$ are adopted and $8\pi G$ is denoted as κ^2. Further the notations for the perturbation variables adopted in the article [1] are used and their definitions are sometimes omitted except for those newly defined in this paper

§2   Review; Evolution of cosmological perturbation in the longwavelength limit

The dynamical behavior of cosmological perturbations on superhorizon scales is well approximated by that of cosmological perturbations in the long wavelength limit [7], and the evolution of cosmological perturbations in the $k \to 0$ limit can be constructed from the perturbation of the exactly homogeneous solution [5]. In this section, we present the results of the paper [5] in case of the universe dominated by multi-component scalar

fields. Next we explain two way to construct the perturbation of the exactly homogeneous solution.

We only consider perturbations on a spatially flat($K = 0$) Robertson-Walker universe throughout the paper. Hence the background metric is given by

$$ds^2 = -dt^2 + a(t)^2 d\boldsymbol{x}^2. \tag{2.1}$$

We consider the universe dominated by multi-component scalar fields, and the energy-momentum tensor is given by

$$T^{\mu}_{\nu} = \nabla^{\mu}\phi \cdot \nabla_{\nu}\phi - \frac{1}{2}\delta^{\mu}_{\nu}\left(\nabla^{\lambda}\phi \cdot \nabla_{\lambda}\phi + 2U\right). \tag{2.2}$$

In the long wavelength limit, the evolution of the gauge invariant variable Y_i representing the fluctuation of the scalar field ϕ_i in the flat time slice are described as

$$
\begin{aligned}
Y_i &= \chi_i + \frac{\dot{\phi}_i}{H}\int dt \frac{H^2}{2U}W\left(\frac{\dot{\phi}}{H}, \chi\right), \\
W(X_1, X_2) &:= X_1 \cdot \dot{X}_2 - \dot{X}_1 \cdot X_2,
\end{aligned}
\tag{2.3}
$$

where χ_i is a combination of exactly homogeneous perturbation variables described as

$$\chi_i = \delta\phi_i - \frac{\dot{\phi}_i}{H}\frac{\delta a}{a}. \tag{2.4}$$

In the $k = 0$ limit, it is noticed that the equality

$$\frac{a^3 H^2}{U}W\left(\frac{\dot{\phi}}{H}, \chi\right) = \text{const} \tag{2.5}$$

holds.

Corresponding to the way how the homogeneous solution is written, we can construct the exactly homogeneous perturbation in the different two ways. One is the way how it is given as derivative of the homogeneous solution with respect to the solution constants. The homogeneous solution is described as

$$\phi_i = \phi_i(t, C), \quad a = a(t, C), \tag{2.6}$$

or

$$\phi_i = \phi_i(a, C), \tag{2.7}$$

where C is the set of all the solution constants characterizing homogeneous solution completely. χ_i is expressed as

$$\chi_i = \frac{\partial \phi_i(t, C)}{\partial C} - \frac{\dot{\phi}_i}{H}\frac{1}{a}\frac{\partial a(t, C)}{\partial C} \tag{2.8}$$

in the above case, and is expressed as

$$\chi_i = \frac{\partial \phi_i(a, C)}{\partial C} \tag{2.9}$$

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in the below case. Noticing the functional relation $a = a(t, C)$, it is seen that the above two expressions are exactly the same. In this paper, the second expression is used when the evolution of the cosmological perturbations in the long wavelength limit is investigated.

On the other hand, the solution is often expressed as union of the manifold on which the first integral is constant

$$F_n(\phi, p_\phi, a) = \text{constant} \quad (n = 1, \cdots, 2N) \tag{2.10}$$

where p_ϕ is canonical momentum conjugate to ϕ as defined later. In this case, the exactly homogeneous perturbation is produced as

$$\chi_i = \{F_n(\phi, p_\phi, a), \phi_i\} = -\frac{\partial F_n(\phi, p_\phi, a)}{\partial p_{\phi_i}}. \tag{2.11}$$

§3   Evolution equations of corresponding exactly homogeneous universe

In this section, we formulate the evolution of the exactly homogeneous universe dominated by multiple scalar fields as a hamiltonian form. Since our system are derived from the general covariant theory, it becomes a constrained system with gauge degree of freedom. By choosing the function of the scale factor as the time variable, we fix the gauge. It is the important first step since the evolution of the cosmological perturbations in the long wavelength limit is generated from the exactly homogeneous perturbations.

We consider the situation where the multiple scalar fields interact with each other

$$S = \int dt(p_a \dot{a} + \sum_i p_{\phi_i} \dot{\phi}_i - NC_H). \tag{3.1}$$

Here N is the lapse function and C_H is the Hamiltonian constraint written as

$$C_H = \Omega a^3 \left(-\frac{\kappa^2}{12} \frac{1}{\Omega^2 a^4} p_a^2 + \frac{1}{2} \frac{1}{\Omega^2 a^6} \sum_i p_{\phi_i}^2 + U(\phi) \right), \tag{3.2}$$

where Ω is the comoving volume of the constant time hypersurface. The potential $U(\phi)$ is decomposed as

$$U(\phi) = \frac{1}{2} \sum_i m_i^2 \phi_i^2 + U_{\text{int}}(\phi). \tag{3.3}$$

where U_{int} is assumed to be a sum of monomials of ϕ of degree not smaller than 3 as in most physical context. We fix the gauge by

$$0 = \psi = a - t, \tag{3.4}$$

and its consistency condition with time evolution determines N as

$$0 = \frac{d\psi}{dt} = -\frac{\kappa^2}{6} \frac{N}{\Omega a} p_a - 1. \tag{3.5}$$

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For notational simplicity we write a in stead of t henceforth. By using Dirac bracket formalism, the Hamiltonian constraint $0 = C_H$ is changed into a strong equation, and p_a can be removed away. We obtain a reduced action with no constraint

$$S = \int \sum_i p_{\phi_i} d\phi_i - h_a da, \tag{3.6}$$

$$h_a = \frac{2\sqrt{3}}{\kappa} \Omega a^2 \Big(\frac{1}{2} \frac{1}{\Omega^2 a^6} \sum_i p_{\phi_i}^2 + U(\phi) \Big)^{1/2}. \tag{3.7}$$

We make all physical quantities non-dimensional by dividing them with typical scales as

$$\zeta = (\frac{a}{a_0})^{3/2}, \tag{3.8}$$

$$\Phi = \frac{\phi}{\phi_0}, \tag{3.9}$$

$$P_\phi = \frac{p_\phi}{a_0^3 m_0 \phi_0 \Omega}, \tag{3.10}$$

$$\mu_i = \frac{m_i}{m_0}, \tag{3.11}$$

$$\epsilon = \frac{\sqrt{3}}{2} \kappa \phi_0, \tag{3.12}$$

$$S' = \frac{S}{\Omega a_0^3 m_0 \phi_0^2},$$

$$= \int \sum_i P_{\phi_i} d\Phi_i$$

$$- \frac{2}{\epsilon} \Big(\frac{1}{2} \frac{1}{\zeta^2} \sum_i P_{\phi_i}^2 + \frac{1}{2} \zeta^2 \sum_i \mu_i^2 \Phi_i^2 + \zeta^2 \frac{1}{m_0^2 \phi_0^2} U_{\text{int}}(\phi_0 \Phi) \Big)^{1/2} d\zeta. \tag{3.13}$$

From now on, we use t instead of ζ without possibility of confusion. In this paper, the evolution of the cosmological perturbations in the long wavelength limit is generated from the derivative of the exactly homogeneous solution with respect to solution constant with the scale factor fixed. Since we adopt the function of the scale factor as the time variable, in order to generate the evolution equations followed by the exactly homogeneous perturbations, it is sufficient to take variations with respect to the canonical variables with the time variable fixed in the unperturbed canonical equations of motion.

§4   Integration Method

This section is the main part of the paper. Since the scalar fields oscillate around the local minimum of the potential, we use the action angle variables, the former of which and the latter of which represent the slowly varying amplitudes and the fast increasing phases of scalar fields oscillation, respectively. We found that our system can be formulated as the canonical perturbation theory the perturbed part of which depending on the angle variables and the negative power of the time appears as a result of the expansion of the universe and the interaction of the scalar fields. Reducing the evolution equation into a simpler

form whose perturbed part is sufficiently small by constructing appropriate coordinate transformations is known as the method of variation of constants mathematically. Since our system is described in the hamiltonian form, we use the canonical transformations. By constructing the canonical transformation properly, we show that the perturbed parts of the hamiltonian can be decreased so that the transformed hamiltonian can be solved easily.

The starting hamiltonian is given by

$$H = \frac{2}{\epsilon}\Big(\frac{1}{2}\frac{1}{t^2}\sum_i P_{\phi_i}^2 + \frac{1}{2}t^2\sum_i \mu_i^2\Phi_i^2 + t^2\frac{1}{m_0^2\phi_0^2}U_{\text{int}}(\phi_0\Phi)\Big)^{1/2}. \tag{4.1}$$

By performing canonical transformation $(\Phi_i, P_{\phi_i}) \rightarrow (\xi^i, \eta^i)$ induced by generating function as

$$W(\Phi,\eta) = \sum_i \sqrt{2}\sqrt{\mu_i}t\eta^i\Phi_i + \sum_i i\mu_i t^2\frac{1}{2}\Phi_i^2 - \sum_i i\frac{1}{2}\eta^{i2}, \tag{4.2}$$

we introduce complex canonical variables as

$$\xi^i = \frac{1}{\sqrt{2}}(\sqrt{\mu_i}t\Phi_i - \frac{1}{\sqrt{\mu_i}}\frac{1}{t}iP_{\phi_i}), \tag{4.3}$$

$$\eta^i = \frac{1}{\sqrt{2}}(-i\sqrt{\mu_i}t\Phi_i + \frac{1}{\sqrt{\mu_i}}\frac{1}{t}P_{\phi_i}). \tag{4.4}$$

Then the transfomed hamiltonian is given by

$$\begin{aligned}H' &= H + \frac{\partial W}{\partial t}\\ &= \frac{2}{\epsilon}\Big(i\sum_i \mu_i\xi^i\eta^i + t^2\frac{1}{m_0^2\phi_0^2}U_{\text{int}}\Big)^{1/2} + \frac{i}{2}\frac{1}{t}\Big(\sum_i \eta^{i2} + \sum_i \xi^{i2}\Big).\end{aligned} \tag{4.5}$$

Since the interaction term U_{int} is written as a sum of monomials of Φ_i;

$$\Phi_i = \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{\mu_i}}\frac{1}{t}\xi^i + i\frac{1}{\sqrt{\mu_i}}\frac{1}{t}\eta^i), \tag{4.6}$$

of degree not smaller than 3, we assume that U_{int} is a sum of monomials in terms of $\zeta^i/t = (\xi^i/t, \eta^i/t)$ of power not smaller than 3 with t-independent coefficients;

$$\frac{t^2}{m_0^2\phi_0^2}U_{\text{int}} = \lambda t^2\sum_{|k|+|l|=3} c_{kl}(\frac{\xi}{t})^k(\frac{\eta}{t})^l + \lambda t^2\sum_{|k|+|l|\geq 4} c_{kl}(\frac{\xi}{t})^k(\frac{\eta}{t})^l, \tag{4.7}$$

where λ is a constant characterizing the strength of interaction. We introduce the action angle variables by

$$\xi^i = \sqrt{J^i}e^{i\theta^i}, \tag{4.8}$$

$$\eta^i = -i\sqrt{J^i}e^{-i\theta^i}. \tag{4.9}$$

Since $k \neq l$ under the condition $|k| + |l| = 3$, the first term of the right hand side K_3;

$$K_3 = \frac{\lambda}{t}k_3(J,\theta), \tag{4.10}$$

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where k_3 is 2π-periodic in $\theta = (\theta^1, \cdots, \theta^N)$ and is of power $3/2$ in terms of J, satisfies

$$\frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} d^N\theta K_3(J, \theta, t) = 0. \tag{4.11}$$

The second term K_4 is written as

$$K_4 = \lambda \sum_{p=2}^{\infty} \frac{1}{t^p} k_{4\ p+2}(J, \theta), \tag{4.12}$$

where $k_{4\ p+2}$ is 2π-periodic in $\theta = (\theta^1, \cdots, \theta^N)$ and is of power $(p+2)/2$ in terms of J. The transformed hamiltonian H' is rewritten as

$$
\begin{aligned}
H &= \frac{2}{\epsilon}(\mu \cdot J + K_3 + K_4)^{1/2} - \sum_i \frac{J^i}{t} \sin 2\theta^i \\
&= \frac{2}{\epsilon}(\mu \cdot J)^{1/2} + \tilde{H}.
\end{aligned}
\tag{4.13}
$$

The interaction part $\tilde{H}$ is decomposed as

$$
\begin{aligned}
\tilde{H} &= \frac{2}{\epsilon}(\mu \cdot J + K_3 + K_4)^{1/2} - \frac{2}{\epsilon}(\mu \cdot J)^{1/2} - \sum_i \frac{J^i}{t} \sin 2\theta^i \\
&= A'_1 + B'_1,
\end{aligned}
\tag{4.14}
$$

where

$$A'_1 = \frac{1}{\epsilon} \frac{1}{(\mu \cdot J)^{1/2}} K_4 + \frac{2}{\epsilon}(\mu \cdot J)^{1/2} \sum_{n=2}^{\infty} \binom{1/2}{n} \left(\frac{K_3 + K_4}{\mu \cdot J}\right)^n, \tag{4.15}$$

$$B'_1 = \frac{1}{\epsilon} \frac{1}{(\mu \cdot J)^{1/2}} K_3 - \sum_i \frac{J^i}{t} \sin 2\theta^i. \tag{4.16}$$

Now we assume that λ/ϵ is at most of order unity. By transferring θ-dependent part of A'_1 to B'_1, we obtain as a starting hamiltonian

$$H = \frac{2}{\epsilon}(\mu \cdot J_1)^{1/2} + A_1(J_1, t) + B_1(J_1, \theta_1, t), \tag{4.17}$$

satisfying (i) $\mu \cdot J_1$ is bounded as

$$|\mu \cdot J_1| \geq \sigma_1, \tag{4.18}$$

for some positive σ_1 on the domain $(J_1,\ 1/t) \in D'_1(\rho_1)$ where

$$D'_n(\rho) = \{(J, \frac{1}{t}); |J - J_{n0}| \leq \rho, |\frac{1}{t}| \leq 1 + \rho, |\mathrm{Im}\frac{1}{t}| \leq \rho\}. \tag{4.19}$$

where J_0 is the initial value of J; $J_0 \equiv J(t=1)$.

(ii) $t^2 A_1$ is analytic in terms of J_1, $1/t$ and is real for real J_1, $1/t$ and the inequality

$$|t^2 A_1| \leq M_1^{(1)} \tag{4.20}$$

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holds for some positive $M_1^{(1)}$ on the domain $D_1(\rho_1)$, where

$$D_n(\rho) = \{(J, \theta, \frac{1}{t}); |J - J_{n0}| \leq \rho, |\mathrm{Im}\theta| \leq \rho, |\frac{1}{t}| \leq 1 + \rho, |\mathrm{Im}\frac{1}{t}| \leq \rho\}. \tag{4.21}$$

(iii) tB_1 is analytic in terms of J_1, θ_1, $1/t$ and is real for real J_1, θ_1, $1/t$ and is 2π-periodic in $\theta_1 = (\theta_1^1, \cdots, \theta_1^N)$ and the inequality

$$|tB_1| \leq M_2^{(1)} \tag{4.22}$$

holds for some positive $M_2^{(1)}$ and satisfy

$$\frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} d^N\theta_1 tB_1(J_1, \theta_1, t) = 0, \tag{4.23}$$

on the domain $D_1(\rho_1)$.

It is noticed that the perturbed parts A_1, B_1 appearing as the result of the expansion of the universe and the interactions of the scalar fields depend on the time as negative power. Since we are interested in the evolution for sufficiently large t, this property plays an essential role in the following analysis.

In reheating phase, scalar fields oscillate rapidly. Since rapid oscillation is diffcult to treat, we construct change of coordinates in order to eliminate fast phases called the angle variables from the transformed equations of perturbed motion. The canonical perturbation theory is effective for this purpose, and can separate the slow secular motions from the fast oscillations by performing the canonical transformations recurrently, when the frequencies of unperturbed motion are incommensurable. With noticing that we can eliminate the dependence on the angle variables from the hamiltonian, it is important to understand that we can increase the power of the inverse of the time $1/t$ which is contained by the perturbed part depending on the angle variables B since we are interested in the evolution for sufficiently large t.

Proposition1 (Theorem for the canonical transformations)

Suppose that $m \geq 1$. Suppose that the hamiltonian

$$H^{(m)} = \frac{2}{\epsilon}(\mu \cdot J_m)^{1/2} + A_m(J_m, t) + B_m(J_m, \theta_m, t) \tag{4.24}$$

satisfy the following conditions:

(i) For some positive C and d, the inequality

$$|k \cdot \mu| \geq C\frac{1}{|k|^d} \tag{4.25}$$

holds for every integral-valued vector $k = (k_1, \cdots, k_N)$.

(ii) $\mu \cdot J_m$ is bounded as

$$|\mu \cdot J_m| \geq \sigma_m, \tag{4.26}$$

for some positive σ_m on the domain $(J_m, 1/t) \in D'_m(\rho_m)$.

(iii) $t^2 A_m(J_m, t)$ is analytic and real for real J_m, $1/t$ and the inequality

$$|t^2 A_m| \leq M_1^{(m)} \tag{4.27}$$

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holds for some positive $M_1^{(m)}$ on the domain $(J_m, 1/t) \in D'_m(\rho_m)$.

(iv) $t^m B_m(J_m, \theta_m, t)$ is periodic in $\theta_m = (\theta_1^{(m)}, \cdots, \theta_N^{(m)})$. It is analytic and real for real J_m, θ_m, $1/t$ and the inequality

$$|t^m B_m| \leq \epsilon^{m-1} M_2^{(m)} \tag{4.28}$$

holds for some positive $M_2^{(m)}$ on the domain $(J_m, \theta_m, 1/t) \in D_m(\rho_m)$ and satisfy

$$\frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} d^N \theta_m t^m B_m(J_m, \theta_m, t) = 0. \tag{4.29}$$

For an arbitrary positive δ, there exists ϵ_0 such that, for an arbitrary ϵ satisfying

$$0 < \epsilon < \epsilon_0, \tag{4.30}$$

there exists the canonical transformation induced by the generating function $S_m(J_{m+1}, \theta_m, t)$ satisfying the following condition;

(v) $t^m S_m(J_{m+1}, \theta_m, t)$ is periodic in $\theta_m = (\theta_1^{(m)}, \cdots, \theta_N^{(m)})$. It is analytic and real for real J_{m+1}, θ_m, $1/t$ and the inequality

$$|t^m S_m| \leq \epsilon^m L_1^{(m)} \tag{4.31}$$

holds for some positive $L_1^{(m)}$ on the domain $(J_{m+1}, \theta_m, 1/t) \in D_{m+1}(\rho_{m+1})$, where $\rho_{m+1} = \rho_m - \delta$.

The transformed system defined by

$$J_m = J_{m+1} + \frac{\partial S_m}{\partial \theta_m}, \tag{4.32}$$

$$\theta_{m+1} = \theta_m + \frac{\partial S_m}{\partial J_{m+1}}, \tag{4.33}$$

$$H^{(m+1)} = H^{(m)} + \frac{\partial S_m}{\partial t} \tag{4.34}$$

$$= \frac{2}{\epsilon}(\mu \cdot J_{m+1})^{1/2} + A_{m+1}(J_{m+1}, t) + B_{m+1}(J_{m+1}, \theta_{m+1}, t) \tag{4.35}$$

has the following properties;

(vi) $\mu \cdot J_{m+1}$ is bounded as

$$|\mu \cdot J_{m+1}| \geq \sigma_{m+1}, \tag{4.36}$$

for some positive σ_{m+1} on the domain $(J_{m+1}, 1/t) \in D'_{m+1}(\rho_{m+1})$

(vii) $t^2 A_{m+1}(J_{m+1}, t)$ is analytic and real for real J_{m+1}, $1/t$ and the inequality

$$|t^2 A_{m+1}| \leq M_1^{(m+1)} \tag{4.37}$$

holds for some positive $M_1^{(m+1)}$ on the domain $(J_{m+1}, 1/t) \in D'_{m+1}(\rho_{m+1})$.

(viii) $t^{m+1} B_{m+1}(J_{m+1}, \theta_{m+1}, t)$ is periodic in $\theta_{m+1} = (\theta_1^{(m+1)}, \cdots, \theta_N^{(m+1)})$. It is analytic and real for real J_{m+1}, θ_{m+1}, $1/t$ and the inequality

$$|t^{m+1} B_{m+1}| \leq \epsilon^m M_2^{(m+1)} \tag{4.38}$$

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holds for some positive $M_2^{(m)}$ on the domain $(J_{m+1}, \theta_{m+1}, 1/t) \in D_{m+1}(\rho_{m+1})$ and satisfy

$$\frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} d^N \theta_{m+1} t^{m+1} B_{m+1}(J_{m+1}, \theta_{m+1}, t) = 0. \tag{4.39}$$

Proof

We consider the canonical transformation induced by the generating function given by

$$S_m(J_{m+1}, \theta_m, t) = \sum_{k \neq 0} S_k(J_{m+1}, t) e^{ik \cdot \theta_m}. \tag{4.40}$$

B_m is decomposed as

$$B_m(J_{m+1}, \theta_m, t) = \sum_{k \neq 0} b_k(J_{m+1}, t) e^{ik \cdot \theta_m}. \tag{4.41}$$

The transformed hamiltonian is

$$
\begin{aligned}
H^{(m+1)} &= H^{(m)} + \frac{\partial S_m}{\partial t} \\
&= \frac{2}{\epsilon}(\mu \cdot J_{m+1})^{1/2} + A_m(J_{m+1}, t) \\
&\quad + \frac{1}{\epsilon} \frac{1}{(\mu \cdot J_{m+1})^{1/2}} \mu \cdot \frac{\partial S_m(J_{m+1}, \theta_m, t)}{\partial \theta_m} + B_m(J_{m+1}, \theta_m, t) \\
&\quad + R_1 + R_2 + R_3 + \frac{\partial S_m(J_{m+1}, \theta_m, t)}{\partial t}
\end{aligned}
\tag{4.42}
$$

where $R_1, \cdots R_3$ are defined as

$$R_1 = \frac{2}{\epsilon}(\mu \cdot J_m)^{1/2} - \frac{2}{\epsilon}(\mu \cdot J_{m+1})^{1/2} - \frac{1}{\epsilon} \frac{1}{(\mu \cdot J_{m+1})^{1/2}} \mu \cdot \frac{\partial S_m(J_{m+1}, \theta_m, t)}{\partial \theta_m}, \tag{4.43}$$

$$R_2 = A_m(J_m, t) - A_m(J_{m+1}, t), \tag{4.44}$$

$$R_3 = B_m(J_m, \theta_m, t) - B_m(J_{m+1}, \theta_m, t). \tag{4.45}$$

We determine the generating function S_m so that the leading term of the oscillating part can be eliminated;

$$\frac{1}{\epsilon} \frac{1}{(\mu \cdot J_{m+1})^{1/2}} \mu \cdot \frac{\partial S_m(J_{m+1}, \theta_m, t)}{\partial \theta_m} = -B_m(J_{m+1}, \theta_m, t). \tag{4.46}$$

By comparing the fourier components in the both side, we obtain

$$S_k(J_{m+1}, t) = i\epsilon(\mu \cdot J_{m+1})^{1/2} \frac{1}{(\mu \cdot k)} b_k(J_{m+1}, t). \tag{4.47}$$

Since from (iv), on the domain $(J_{m+1}, 1/t) \in D'_m(\rho_m)$, the inequality

$$|t^m b_k(J_{m+1}, t)| \leq \epsilon^{m-1} M_2^{(m)} \exp(-|k|\rho_{m-1}) \tag{4.48}$$

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holds, we evaluate the upper bound of the fourier components S_k;

$$
\begin{aligned}
|t^m S_k(J_{m+1}, t)| &\leq \epsilon(\mu \cdot J_{m0} + N\mu\rho_m)^{1/2} \frac{|k|^d}{C} \epsilon^{m-1} M_2^{(m)} \exp(-|k|\rho_m) \\
&\leq \epsilon^m (\mu \cdot J_{m0} + N\mu\rho_m)^{1/2} \frac{M_2^{(m)}}{C} (\frac{d}{e})^d \frac{1}{\delta^d} \exp\{-|k|(\rho_m - \delta)\}. \quad (4.49)
\end{aligned}
$$

where in proceeding from the first row to the second row we use the inequality

$$
|k|^d \leq (\frac{d}{e})^d \frac{\exp(|k|\delta)}{\delta^d}, \qquad (4.50)
$$

for arbitrary positive $|k|$, d, δ. On the domain $|J_{m+1} - J_{m0}| \leq \rho_m$, $|\mathrm{Im}\theta_m| \leq \rho_m - 2\delta$, $t^m S_m$ is bounded as

$$
\begin{aligned}
|t^m S_m| &\leq \sum_{k \neq 0} |t^m S_k| \exp(|k||\mathrm{Im}\theta_m|) \\
&\leq \epsilon^m (\mu \cdot J_{m0} + N\mu\rho_m)^{1/2} \frac{M_2^{(m)}}{C} (\frac{d}{e})^d \frac{1}{\delta^d} \sum_{k \neq 0} \exp(-|k|\delta) \\
&\leq \epsilon^m L_1^{(m)}, \qquad (4.51)
\end{aligned}
$$

where $L_1^{(m)}$ is defined as

$$
L_1^{(m)} = (\mu \cdot J_{m0} + N\mu\rho_m)^{1/2} \frac{M_2^{(m)}}{C} (\frac{d}{e})^d \frac{1}{\delta^d} (\frac{4}{\delta})^N \qquad (4.52)
$$

In passing from the second row to the third row, we use the inequality

$$
\begin{aligned}
\sum_{k \neq 0} \exp(-|k|\delta) &< [1 + 2\sum_{k>0} \exp(-k\delta)]^N \\
&< (\frac{4}{\delta})^N, \qquad (4.53)
\end{aligned}
$$

where we assume that $\delta < 1$. On the domain $|J_{m+1} - J_{m0}| \leq \rho_m$, $|\mathrm{Im}\theta_m| \leq \rho_m - 3\delta$, $|1/t| \leq 1 + \rho_m$, $t^m \partial S_m / \partial \theta_m$ is bounded as

$$
|t^m \frac{\partial S_m(J_{m+1}, \theta_m, t)}{\partial \theta_m}| \leq \epsilon^m \frac{L_1^{(m)}}{\delta}. \qquad (4.54)
$$

So the inequality

$$
|J_{m0} - J_{m+1\,0}| = |\frac{\partial S_m(J_{m+1\,0}, \theta_{m\,0}, t=1)}{\partial \theta_{m\,0}}| \leq \epsilon^m \frac{L_1^{(m)}}{\delta} \qquad (4.55)
$$

holds. Of course, the above two inequalities hold on the domain $|J_{m+1} - J_{m0}| \leq \rho_m - 3\delta$, $|\mathrm{Im}\theta_m| \leq \rho_m - 3\delta$, $|1/t| \leq 1 + \rho_m$. Then

$$
\begin{aligned}
|J_{m+1} - J_{m+1\,0}| &\leq |J_{m+1} - J_{m0}| + |J_{m0} - J_{m+1\,0}| \\
&\leq \rho_m - 3\delta + \epsilon^m \frac{L_1^{(m)}}{\delta}. \qquad (4.56)
\end{aligned}
$$

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When we assume that ϵ satisfies

$$\epsilon^m \frac{L_1^{(m)}}{\delta} \leq \delta, \tag{4.57}$$

the inequality

$$|J_{m+1} - J_{m+1\ 0}| \leq \rho_m - 2\delta \tag{4.58}$$

holds. The inequality (4.51) holds on the domain $(J_{m+1}, \theta_m, 1/t) \in D(\rho_m - 2\delta)$.

Now for an arbitrarily fixed J_{m+1} on the domain $|J_{m+1} - J_{m+1\ 0}| \leq \rho_m - \delta$, we consider the mapping $T_{J_{m+1}}$ defined by

$$T_{J_{m+1}}; \theta_m \to \theta_{m+1} = \theta_m + \frac{\partial S_m}{\partial J_{m+1}}(J_{m+1}, \theta_m, t) \tag{4.59}$$

For an arbitrary θ_m on the domain $|\mathrm{Im}\theta_m| \leq \rho_m - 2\delta$, the inequality

$$
\begin{aligned}
|\theta_m - \theta_{m+1}| &\leq |\frac{\partial S_m}{\partial J_{m+1}}(J_{m+1}, \theta_m, t)| \\
&\leq \epsilon^m (1 + \rho_m)^m \frac{L_1^{(m)}}{\delta}
\end{aligned}
\tag{4.60}
$$

holds. Therefore the mapping $T_{J_{m+1}}$ is injective (so diffeomorphic) on the domain $|\mathrm{Im}\theta_m| \leq \rho_m - 2\delta - 4\epsilon^m(1 + \rho_m)^m L_1^{(m)}/\delta$. $T_{J_{m+1}}$ maps an arbitrary θ_m on the domain $|\mathrm{Im}\theta_m| \leq \rho_m - 2\delta - 4\epsilon^m(1 + \rho_m)^m L_1^{(m)}/\delta$ to θ_{m+1} on the domain $|\mathrm{Im}\theta_{m+1}| \leq \rho_m - 2\delta - 3\epsilon^m(1 + \rho_m)^m L_1^{(m)}/\delta$. So on the domain $|J_{m+1} - J_{m+1\ 0}| \leq \rho_m - \delta$, $|\mathrm{Im}\theta_{m+1}| \leq \rho_m - 2\delta - 3\epsilon^m(1 + \rho_m)^m L_1^{(m)}/\delta$, the inverse mapping $T_{J_{m+1}}^{-1}$ is defined and diffeomorphic and $|\mathrm{Im}T_{J_{m+1}}^{-1}\theta_{m+1}| = |\mathrm{Im}\theta_m| \leq \rho_m - 2\delta - 4\epsilon^m(1 + \rho_m)^m L_1^{(m)}/\delta$. From now on, we assume that ϵ satisfies

$$3\epsilon^m(1 + \rho_m)^m \frac{L_1^{(m)}}{\delta} \leq \delta. \tag{4.61}$$

Then the canonical transformation $(J_{m+1}, \theta_{m+1}) \to (J_m, \theta_m)$ induced by the generating function $S_m(J_{m+1}, \theta_m, t)$ is defined and is diffeomorphic on the domain $|J_{m+1} - J_{m+1\ 0}| \leq \rho_m - \delta$, $|\mathrm{Im}\theta_m| \leq \rho_m - 3\delta$ and $|\mathrm{Im}T_{J_{m+1}}^{-1}\theta_{m+1}| = |\mathrm{Im}\theta_m| \leq \rho_m - 3\delta + \delta/3 \leq \rho_m - 2\delta$.

Next on the domain $|J_{m+1} - J_{m+1\ 0}| \leq \rho_m - 2\delta$, $|\mathrm{Im}\theta_m| \leq \rho_m - 3\delta$, we evaluate the residual part R defined as

$$R = R_1 + R_2 + R_3 + \frac{\partial S_m(J_{m+1}, \theta_m, t)}{\partial t}. \tag{4.62}$$

R_1 can be rewritten as

$$
\begin{aligned}
R_1 &= \int_{J_{m+1}}^{J_m} \mu \cdot dJ \frac{1}{\epsilon}\{\frac{1}{(\mu \cdot J)^{1/2}} - \frac{1}{(\mu \cdot J_{m+1})^{1/2}}\} \\
&= -\int_{J_{m+1}}^{J_m} \mu \cdot dJ \frac{1}{2\epsilon}\{\int_{J_{m+1}} \mu \cdot dJ \frac{1}{(\mu \cdot J)^{3/2}}\}.
\end{aligned}
\tag{4.63}
$$

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We evaluate

$$
\begin{aligned}
|t^{2m}R_1| &\leq \epsilon^{2m-1}\mu^2 N^2 (1+\rho_m)^{2m}\frac{L_1^{(m)2}}{\delta^2}\frac{1}{\sigma_m^{3/2}} \\
&\leq \epsilon^m \mu^2 N^2 (1+\rho_m)^{2m}\frac{L_1^{(m)2}}{\delta^2}\frac{1}{\sigma_m^{3/2}}.
\end{aligned}
\tag{4.64}
$$

Therefore we obtain

$$
|t^{m+1}R_1| \leq \epsilon^m \mu^2 N^2 (1+\rho_m)^{3m-1}\frac{L_1^{(m)2}}{\delta^2}\frac{1}{\sigma_m^{3/2}}.
\tag{4.65}
$$

As for R_2, we obtain

$$
\begin{aligned}
|t^{m+2}R_2| &= |t^{m+2}\int_{J_{m+1}}^{J_m}\frac{\partial A_m(J_m,t)}{\partial J_m}dJ_m| \\
&\leq \sup\{|t^2\frac{\partial A_m(J_m,t)}{\partial J_m}|\}|t^m\frac{\partial S_m(J_{m+1},\theta_m,t)}{\partial \theta_m}| \\
&\leq \epsilon^m\frac{M_1^{(m)}L_1^{(m)}}{6\delta^2}.
\end{aligned}
\tag{4.66}
$$

Therefore we obtain

$$
|t^{m+1}R_2| \leq \epsilon^m (1+\rho_m)\frac{M_1^{(m)}L_1^{(m)}}{6\delta^2}.
\tag{4.67}
$$

As for R_3, we obtain

$$
\begin{aligned}
|t^{2m}R_3| &= |t^{2m}\int_{J_{m+1}}^{J_m}\frac{\partial B_m(J_m,\theta_m,t)}{\partial J_m}dJ_m| \\
&\leq \sup\{|t^m\frac{\partial B_m(J_m,\theta_m,t)}{\partial J_m}|\}|t^m\frac{\partial S_m(J_{m+1},\theta_m,t)}{\partial \theta_m}| \\
&\leq \epsilon^{2m-1}\frac{M_2^{(m)}L_1^{(m)}}{6\delta^2} \\
&\leq \epsilon^m\frac{M_2^{(m)}L_1^{(m)}}{6\delta^2}
\end{aligned}
\tag{4.68}
$$

Therefore we obtain

$$
|t^{m+1}R_3| \leq \epsilon^m (1+\rho_m)^{m-1}\frac{M_2^{(m)}L_1^{(m)}}{6\delta^2}.
\tag{4.69}
$$

Also we obtain

$$
|t^{m+1}\frac{\partial S_m}{\partial t}| \leq \epsilon^m[(m-1)L_1^{(m)}+(1+\rho_m)\frac{L_1^{(m)}}{\delta}].
\tag{4.70}
$$

So R is evaluated as

$$
|t^{m+1}R| \leq \epsilon^m M,
\tag{4.71}
$$

where

$$M = \mu^2 N^2 (1+\rho_m)^{3m-1} \frac{L_1^{(m)2}}{\delta^2} \frac{1}{\sigma_m^{3/2}} + (1+\rho_m) \frac{M_1^{(m)} L_1^{(m)}}{6\delta^2} + (1+\rho_m)^{m-1} \frac{M_2^{(m)} L_1^{(m)}}{6\delta^2}$$

$$+(m-1)L_1^{(m)} + (1+\rho_m)\frac{L_1^{(m)}}{\delta}, \tag{4.72}$$

on the domain $(J_{m+1}, \theta_{m+1}, 1/t) \in D_{m+1}(\rho_m - 2\delta)$. We decompose R into the fourier components as

$$R(J_{m+1}, \theta_{m+1}, t) = \sum_k R_k(J_{m+1}, t) e^{ik\theta_{m+1}}. \tag{4.73}$$

As for A_{m+1}, B_{m+1} defined as

$$A_{m+1}(J_{m+1}, t) = A_m(J_{m+1}, t) + R_0(J_{m+1}, t), \tag{4.74}$$

$$B_{m+1}(J_{m+1}, \theta_{m+1}, t) = \sum_{k \neq 0} R_k(J_{m+1}, t) e^{ik\theta_{m+1}}, \tag{4.75}$$

we obtain

$$|t^2 A_{m+1}| \le M_1^{(m+1)}, \tag{4.76}$$

$$|t^{m+1} B_{m+1}| \le \epsilon^m M_2^{(m+1)}, \tag{4.77}$$

where

$$M_1^{(m+1)} = M_1^{(m)} + (1+\rho_m)^{m-2} \epsilon^m M, \tag{4.78}$$

$$M_2^{(m+1)} = 2M \tag{4.79}$$

on the domain $(J_{m+1}, \theta_{m+1}, 1/t) \in D_{m+1}(\rho_m - 2\delta)$. We put $\rho_m = \rho_{m-1} - 2\delta$. We complete the proof.

Proof End

The following proposition is used in evaluating the difference between the original perturbation variables and the transformed perturbation variables. Since as shown later, the action variable perturbations and the angle variable perturbations are bounded by t^0, t respectively, the difference between the original perturbation variables and the transformed perturbation variables is bounded by some positive constant.

Proposition2

Suppose that the generating functions S_m are analytic and the inequalities

$$|t^m S_m| \le \epsilon^m L_1^{(m)}, \tag{4.80}$$

$$|t^m (\frac{\partial}{\partial J_{m+1}})^{\alpha_1} (\frac{\partial}{\partial \theta_m})^{\alpha_2} S_m| \le \epsilon^m L_2^{(m)} \quad (\alpha_1 + \alpha_2 \le 2) \tag{4.81}$$

holds for some positive $L_1^{(m)}$, $L_2^{(m)}$.

As for $m \geq 2$, the inequalities

$$|\delta J_1^i(t) - \delta J_m^i(t)| \leq \frac{\epsilon}{t} C_{11}^{(m)} |\delta J_m(t)| + \frac{\epsilon}{t} C_{12}^{(m)} |\delta \theta_m(t)|, \tag{4.82}$$

$$|\delta \theta_1^i(t) - \delta \theta_m^i(t)| \leq \frac{\epsilon}{t} C_{21}^{(m)} |\delta J_m(t)| + \frac{\epsilon}{t} C_{22}^{(m)} |\delta \theta_m(t)| \tag{4.83}$$

holds for some positive $C_{11}^{(m)}$, $C_{12}^{(m)}$, $C_{21}^{(m)}$, $C_{22}^{(m)}$, where

$$|\delta J_m(t)| \equiv \max_{1 \leq i \leq N} |\delta J_m^i(t)|, \tag{4.84}$$

$$|\delta \theta_m(t)| \equiv \max_{1 \leq i \leq N} |\delta \theta_m^i(t)|. \tag{4.85}$$

Proof

By taking the variation of

$$J_m^i - J_{m+1}^i = \frac{\partial S_m}{\partial \theta_m^i}, \qquad \theta_{m+1}^i - \theta_m^i = \frac{\partial S_m}{\partial J_{m+1}^i}, \tag{4.86}$$

we obtain

$$|\delta J_m^i - \delta J_{m+1}^i| \leq \frac{\epsilon^m}{t^m} N L_2^{(m)} (|\delta J_{m+1}| + |\delta \theta_m|), \tag{4.87}$$

$$|\delta \theta_{m+1}^i - \delta \theta_m^i| \leq \frac{\epsilon^m}{t^m} N L_2^{(m)} (|\delta J_{m+1}| + |\delta \theta_m|). \tag{4.88}$$

As for $|\delta \theta_m^i|$, we obtain

$$
\begin{aligned}
|\delta \theta_m^i| &\leq |\delta \theta_m^i - \delta \theta_{m+1}^i| + |\delta \theta_{m+1}^i| \\
&\leq \frac{\epsilon^m}{t^m} N L_2^{(m)} (|\delta J_{m+1}| + |\delta \theta_m|) + |\delta \theta_{m+1}|,
\end{aligned} \tag{4.89}
$$

which yield

$$|\delta \theta_m| \leq \frac{\epsilon^m}{t^m} N L_2^{(m)'} |\delta J_{m+1}| + (1 + \epsilon^{m-1} N L_2^{(m)''}) |\delta \theta_{m+1}|. \tag{4.90}$$

By using the above inequality, the inequalities (4.87), (4.88) are rewritten as

$$|\delta J_m^i - \delta J_{m+1}^i| \leq \frac{\epsilon^m}{t^m} N L_2^{(m)'''} (|\delta J_{m+1}| + |\delta \theta_{m+1}|), \tag{4.91}$$

$$|\delta \theta_{m+1}^i - \delta \theta_m^i| \leq \frac{\epsilon^m}{t^m} N L_2^{(m)'''} (|\delta J_{m+1}| + |\delta \theta_{m+1}|). \tag{4.92}$$

By using the above inequalities, by the induction, it is shown that for an arbitrary natural number $m \geq 2$, the inequalities

$$|\delta J_1^i(t) - \delta J_m^i(t)| \leq \frac{\epsilon}{t} C_{11}^{(m)} |\delta J_m(t)| + \frac{\epsilon}{t} C_{12}^{(m)} |\delta \theta_m(t)|, \tag{4.93}$$

$$|\delta \theta_1^i(t) - \delta \theta_m^i(t)| \leq \frac{\epsilon}{t} C_{21}^{(m)} |\delta J_m(t)| + \frac{\epsilon}{t} C_{22}^{(m)} |\delta \theta_m(t)| \tag{4.94}$$

holds for some positive $C_{11}^{(m)}$, $C_{12}^{(m)}$, $C_{21}^{(m)}$, $C_{22}^{(m)}$.

Proof End

§5 Evolution of the exactly homogeneous universe

In this section, we investigate the evolutionary behavior of the exactly homogeneous universe dominated by scalar fields. In terms of action angle variables, the evolution of the scalar fields is characterized by the perpetual stability of the action variables which we formulate below.

Proposition3 (Perpetual stability of the action variables)

Suppose that the hamiltonian

$$H = \frac{2}{\epsilon}\Big(\frac{1}{2}\frac{1}{t^2}\sum_i P_{\phi_i}^2 + \frac{1}{2}t^2\sum_i \mu_i^2\Phi_i^2 + \lambda t^2 U_{\text{int}}(\Phi)\Big)^{1/2} \tag{5.1}$$

which satisfy the following conditions;

(i) For some positive C and d, the inequality

$$|k \cdot \mu| \geq C\frac{1}{|k|^d} \tag{5.2}$$

holds for every integral-valued vector $k = (k_1, \cdots, k_N)$.

In terms of action angle variables introduced by

$$\Phi_i \;=\; \sqrt{\frac{2}{\mu_i}}\sqrt{J_i}\frac{1}{t}\cos\theta_i, \tag{5.3}$$

$$P_{\phi_i} \;=\; -\sqrt{2\mu_i}\sqrt{J_i}t\sin\theta_i, \tag{5.4}$$

(ii)$U_{\text{int}}(\Phi)$ is a sum of monomials in terms of Φ of power not smaller than 3. The inequality

$$|\lambda t^2 U_{\text{int}}(\Phi)| \leq \epsilon M \tag{5.5}$$

holds for some positive M on the domain $(J, \theta, 1/t) \in D(\rho)$

(iii) The inequality

$$|\frac{1}{2}\frac{1}{t^2}\sum_i P_{\phi_i}^2 + \frac{1}{2}t^2\sum_i \mu_i^2\Phi_i^2| = |\mu \cdot J| > \sigma \tag{5.6}$$

holds for some positive σ on the domain $(J, 1/t) \in D'(\rho)$.

There exists some positive ϵ_0 such that for an arbitrary ϵ satisfying

$$0 < \epsilon < \epsilon_0, \tag{5.7}$$

the inequality

$$|J - J_0| \leq \epsilon C \tag{5.8}$$

holds for some positive C.

Proof

We consider the sequence of canonical transformations defined in the **Proposition1**.

$$H^{(1)}(J_1, \theta_1, t) \xrightarrow{S_1} H^{(2)}(J_2, \theta_2, t). \tag{5.9}$$

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The transformed hamiltonian is given by

$$H^{(2)} = \frac{2}{\epsilon}(\mu \cdot J_2)^{1/2} + A_2(J_2, t) + B_2(J_2, \theta_2, t), \tag{5.10}$$

where

$$\frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} d^N\theta_3 t^2 B_2(J_2, \theta_2, t) = 0. \tag{5.11}$$

The inequality

$$|t^2 A_2| \le M_1^{(2)}, \quad |t^2 B_2| \le \epsilon M_2^{(2)} \tag{5.12}$$

is satisfied on the domain $D_2(\rho_2)$. The equation of J_2 is

$$\frac{dJ_2^i}{dt} = -\frac{\partial B_2}{\partial \theta_2^i}. \tag{5.13}$$

For an arbitrary positive δ, on the domain $D(\rho_2 - \delta)$

$$\begin{aligned}
|J_2^i(t) - J_2^i{}_0| &\le \int_1 dt |\frac{\partial B_2}{\partial \theta_2^i}| \\
&\le \int_1 dt \frac{\epsilon}{t^2} \frac{M_2^{(2)}}{\delta} \\
&\le \epsilon \frac{M_2^{(2)}}{\delta}.
\end{aligned} \tag{5.14}$$

for an arbitrary $t \ge 1$.

For an arbitrary positive ϵ satisfying

$$\sqrt{N}\epsilon \frac{M_2^{(2)}}{\delta} \le \rho_2 - \delta \tag{5.15}$$

the inequality

$$|J_2 - J_{20}| \le \sqrt{N}\epsilon \frac{M_2^{(2)}}{\delta}. \tag{5.16}$$

holds.

We complete the proof.

Proof End

§6   Evolution of Cosmological Perturbations in the Long Wavelength Limit

We consider the evolution of the cosmological perturbations in the long wavelength limit since it approximates the evolution of the cosmological perturbations on superhorizon scales well [7]. In this section, by using the method of the long wavelength limit, and the canonical perturbation theory, we investigated the evolutionary behavior of the cosmological perturbations, in particular paying attention to that of the Bardeen parameter Z. We

show that for the adiabatic growing mode Z stays constant, and for all the other modes Z decays. Since the Bardeen parameter becomes singular probabilistically when the masses of the scalar fields are incommensurable, we discuss the distribution of the singularities of Z in the time axis in terms of the ergodic theory.

On superhorizon scales, the Bardeen parameter defined by

$$Z = \mathcal{R} - \frac{aH}{k}\sigma_g,$$
(6.1)

is conserved in a good accuracy when the equation of state is regular and the energy of the universe is dominated by a single component. On the other hand, in the universe dominated by multi-component scalar fields, the Bardeen parameter is rewritten as

$$Z = -H\frac{\dot{\phi}\cdot Y}{(\dot{\phi})^2},$$
(6.2)

whose conservation is not self-evident. By using Eq.(2.5), we can see that the contribution from the second term in the right hand side of Eq.(2.3) is given by

$$Z \propto \int dt\frac{1}{a^3}.$$
(6.3)

So we concentrate the contribution from the first term in the right hand side of Eq.(2.3) henceforth.

$$Z = -H\frac{\dot{\phi}\cdot\delta\phi}{(\dot{\phi})^2}.$$
(6.4)

The contribution to the Bardeen parameter from the exactly homogeneous perturbation, Eq.(6.4), is rewritten in terms of the action angle variables as

$$
\begin{aligned}
Z &= \frac{2}{3}\frac{\epsilon}{t}(\mu\cdot J + K)^{1/2}\frac{1}{\sum_i \mu_i J^i(1-\cos 2\theta^i)} \\
&\quad \sum_i\left(\frac{1}{2}\delta J^i\sin 2\theta^i - J^i(1-\cos 2\theta^i)\delta\theta^i\right).
\end{aligned}
$$
(6.5)

In order to understand the evolutionary behavior of the perturbation variables such as Z, we have to investigate that of $\delta\theta^i$, δJ^i. Since the oscillating part is troublesome for investigating the differential equations of perturbations, we eliminate the fast phases by the canonical transformations. We consider the sequence of canonical transformations defined in the **Propositions1**.

$$H^{(1)}(J_1,\theta_1,t) \xrightarrow{S_1} H^{(2)}(J_2,\theta_2,t) \xrightarrow{S_2} H^{(3)}(J_3,\theta_3,t).$$
(6.6)

The transformed hamiltonian is given by

$$H^{(3)} = \frac{2}{\epsilon}(\mu\cdot J_3)^{1/2} + A_3(J_3,t) + B_3(J_3,\theta_3,t),$$
(6.7)

where

$$|t^2 A_3| \le M_1^{(3)}, \quad |t^3 B_3| \le \epsilon^2 M_2^{(3)}$$
(6.8)

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is satisfied on the domain $D_3(\rho_3)$. The variational equation of motion changed into the integral form is

$$\delta J_3^i = \delta J_{30}^i - \int_1 dt [\frac{\partial^2 B_3}{\partial \theta_3^i \partial \theta_3} \delta \theta_3 + \frac{\partial^2 B_3}{\partial \theta_3^i \partial J_3} \delta J_3], \tag{6.9}$$

$$\frac{\delta \theta_3^i}{t} = \frac{\delta \theta_{30}^i}{t} + \frac{1}{t} \frac{\mu_i}{\epsilon} \int_1 dt \delta \frac{1}{(\mu \cdot J_3)^{1/2}}$$
$$+ \frac{1}{t} \int_1 dt \frac{\partial^2 A_3}{\partial J_3^i \partial J_3} \delta J_3 + \frac{1}{t} \int_1 dt [\frac{\partial^2 B_3}{\partial J_3^i \partial \theta_3} \delta \theta_3 + \frac{\partial^2 B_3}{\partial J_3^i \partial J_3} \delta J_3]. \tag{6.10}$$

For an arbitrary positive δ, on the domain $D_3(\rho_3 - \delta)$, we obtain

$$|\delta J_3^i(t)| \leq |\delta J_{30}^i| + \frac{2}{\delta^2} \epsilon^2 M_2^{(3)} \{|\frac{\delta \theta_3}{t}|_m(t) + \frac{1}{2}|\delta J_3|_m(t)\}, \tag{6.11}$$

$$|\frac{\delta \theta_3^i(t)}{t}| \leq |\delta \theta_{30}^i| + \frac{\mu_i}{\epsilon} \frac{1}{2} \frac{N\mu}{\sigma_3^{3/2}} |\delta J_3|_m(t) + \frac{1}{t} \frac{2}{\delta^2} M_1^{(3)} |\delta J_3|_m(t)$$
$$+ \frac{1}{t} \frac{2}{\delta^2} \epsilon^2 M_2^{(3)} \{|\frac{\delta \theta_3}{t}|_m(t) + \frac{1}{2}|\delta J_3|_m(t)\}, \tag{6.12}$$

where

$$|A|_m(t) = \sup_{1 \leq s \leq t} |A(s)| \tag{6.13}$$

They yield

$$|\delta J_3|_m(t) \leq (1 + \epsilon C_1)|\delta J_{30}| + \epsilon^2 C_2 |\delta \theta_{30}|, \tag{6.14}$$

$$|\frac{\delta \theta_3}{t}|_m(t) \leq \frac{1}{\epsilon} C_3 |\delta J_{30}| + (1 + \epsilon C_4)|\delta \theta_{30}|, \tag{6.15}$$

for some positive $C_1, \cdots C_4$. In particular, when Eq.(6.10) is written as

$$\delta \theta_3^i = \delta \theta_{30}^i + \frac{\mu_i}{\epsilon}(t - 1)V_3(t) + W_3^i(t), \tag{6.16}$$

$$V_3(t) = \frac{1}{(t - 1)} \int_1 dt \delta \frac{1}{(\mu \cdot J_3)^{1/2}}, \tag{6.17}$$

$$W_3^i(t) = \int_1 dt \frac{\partial^2 A_3}{\partial J_3^i \partial J_3} \delta J_3 + \int_1 dt [\frac{\partial^2 B_3}{\partial J_3^i \partial \theta_3} \delta \theta_3 + \frac{\partial^2 B_3}{\partial J_3^i \partial J_3} \delta J_3], \tag{6.18}$$

from the inequalities (6.14), (6.15), we obtain

$$|V_3(t) - \delta \frac{1}{(\mu \cdot J_{30})^{1/2}}| \leq \epsilon(C_5 |\delta J_{30}| + \epsilon C_6 |\delta \theta_{30}|), \tag{6.19}$$

$$|W_3^i(t)| \leq C_7 |\delta J_{30}| + \epsilon^2 C_8 |\delta \theta_{30}|. \tag{6.20}$$

Next we have to know the difference between the original variables $\delta \theta_1$, δJ_1 and the transformed variables $\delta \theta_3$, δJ_3, since we want to know the information of the original variables. As for the initial values between of $\delta \theta_1$, δJ_1 and of $\delta \theta_m$, δJ_m, the inequalities

$$|\delta J_{10}^i - \delta J_{m0}^i| \leq \epsilon C'^{(m)}_{11} |\delta J_{10}| + \epsilon C'^{(m)}_{12} |\delta \theta_{10}|, \tag{6.21}$$

$$|\delta \theta_{10}^i - \delta \theta_{m0}^i| \leq \epsilon C'^{(m)}_{21} |\delta J_{10}| + \epsilon C'^{(m)}_{22} |\delta \theta_{10}|. \tag{6.22}$$

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and the inequalities

$$|\delta J_{m0}| \leq (1 + \epsilon C_{33}^{(m)})|\delta J_{10}| + \epsilon C_{34}^{(m)}|\delta \theta_{10}|, \tag{6.23}$$

$$|\delta \theta_{m0}| \leq \epsilon C_{43}^{(m)}|\delta J_{10}| + (1 + \epsilon C_{44}^{(m)})|\delta \theta_{10}|, \tag{6.24}$$

hold. Using (6.23), (6.24) to (6.14), (6.15) we obtain

$$|\delta J_3|_m(t) \leq (1 + \epsilon C_1^{(4)})|\delta J_{10}| + \epsilon C_2^{(4)}|\delta \theta_{10}|, \tag{6.25}$$

$$|\frac{\delta \theta_3}{t}|_m(t) \leq \frac{1}{\epsilon} C_3^{(4)}|\delta J_{10}| + C_4^{(4)}|\delta \theta_{10}|. \tag{6.26}$$

By using (6.25), (6.26) to (4.82), (4.83), we obtain

$$|\delta J_1^i(t) - \delta J_3^i(t)| \leq C_1|\delta J_{10}| + \epsilon C_2|\delta \theta_{10}|, \tag{6.27}$$

$$|\delta \theta_1^i(t) - \delta \theta_3^i(t)| \leq C_3|\delta J_{10}| + \epsilon C_4|\delta \theta_{10}|. \tag{6.28}$$

From (6.25), (6.27), we obtain

$$|\delta J_1^i(t)| \leq C_5|\delta J_{10}| + \epsilon C_6|\delta \theta_{10}|. \tag{6.29}$$

This yields the evaluation (6.37).

On the other hand, from (6.19), (6.20), (6.23), (6.24), (6.28), we obtain the evaluation (6.37), (6.38), $\cdots$ (6.40).

Proposition4 (Evaluation of the perturbation variables)

Suppose that the hamiltonian

$$H = \frac{2}{\epsilon} \left(\frac{1}{2} \frac{1}{t^2} \sum_i P_{\phi_i}^2 + \frac{1}{2} t^2 \sum_i \mu_i^2 \Phi_i^2 + \lambda t^2 U_{\text{int}}(\Phi) \right)^{1/2} \tag{6.30}$$

which satisfy the following conditions;

(i) For some positive C and d, the inequality

$$|k \cdot \mu| \geq C \frac{1}{|k|^d} \tag{6.31}$$

holds for every integral-valued vector $k = (k_1, \cdots, k_N)$.

In terms of action angle variables introduced by

$$\Phi_i = \sqrt{\frac{2}{\mu_i}} \sqrt{J_i} \frac{1}{t} \cos \theta_i, \tag{6.32}$$

$$P_{\phi_i} = -\sqrt{2\mu_i} \sqrt{J_i} t \sin \theta_i, \tag{6.33}$$

(ii) $U_{\text{int}}(\Phi)$ is a sum of monomials in terms of Φ of power not smaller than 3. The inequality

$$|\lambda t^2 U_{\text{int}}(\Phi)| \leq \epsilon M \tag{6.34}$$

holds for some positive M on the domain $(J, \theta, 1/t) \in D(\rho)$.

(iii) The inequality

$$|\frac{1}{2}\frac{1}{t^2}\sum_i P_{\phi_i}^2 + \frac{1}{2}t^2\sum_i \mu_i^2\Phi_i^2| = |\mu\cdot J| > \sigma \qquad (6.35)$$

holds for some positive σ on the domain $(J, 1/t) \in D'(\rho)$

There exists some positive ϵ_0 such that;

for an arbitrary ϵ satisfying

$$0 < \epsilon < \epsilon_0, \qquad (6.36)$$

the following inequalities holds.

As for $\delta J^i(t)$, the inequality

$$|\delta J^i(t) - \delta J_0^i| \le C_1|\delta J_0| + \epsilon C_2|\delta\theta_0| \qquad (6.37)$$

holds for some positive C_1, C_2.

When $\delta\theta^i(t)$ is decomposed as

$$\delta\theta^i(t) = \delta\theta_0^i + \frac{\mu_i}{\epsilon}(t-1)V(t) + W^i(t). \qquad (6.38)$$

(i) $V(t)$ is i-indendent and the inequality

$$|V(t) - \delta(\frac{1}{\mu\cdot J_0})| \le \epsilon(C_3|\delta J_0| + \epsilon C_4|\delta\theta_0|) \qquad (6.39)$$

holds for some positive C_3 and C_4.

(ii) $W^i(t)$ is i-dependent in general and the inequality

$$|W^i(t)| \le C_5|\delta J_0| + \epsilon C_6|\delta\theta_0| \qquad (6.40)$$

holds for some positive C_5 and C_6.

The decomposition of $\delta\theta^i(t)$ into $V(t)$ and $W^i(t)$ as in (6.38) is not unique. For the purpose explained below, it may be rather convenient to write in the next form.

Proposition4'

Under the same assumption presented in the above proposition, for $t \ge 2$, there exists a function $V(t)$ such that

$$\delta\theta^i(t) = \delta\theta_0^i + \frac{\mu_i}{\epsilon}(t-1)V(t) + W^i(t), \qquad (6.41)$$

$$|V(t) - V(t=\infty)| \le \frac{\epsilon^2}{t}(C_1|\delta J_0| + \epsilon C_2|\delta\theta_0|), \qquad (6.42)$$

$$|W^i(t)| \le C_3|\delta J_0| + \epsilon C_4|\delta\theta_0|, \qquad (6.43)$$

for some positive $C_1, \cdots, C_4$.

Proof

We consider the transformed hamiltonian given by

$$H^{(4)} = \frac{2}{\epsilon}(\mu\cdot J_4)^{1/2} + A_4(J_4, t) + B_4(J_4, \theta_4, t), \qquad (6.44)$$

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where
$$|t^2 A_4| \le M_1^{(4)}, \quad |t^4 B_4| \le \epsilon^3 M_2^{(4)} \tag{6.45}$$

is satisfied on the domain $D_4(\rho_4)$. As for $J_4^i(t)$ and $\delta J_4^i(t)$, we obtain inequalities

$$
\begin{aligned}
|J_4^i(t) - J_4^i{}_\infty| &\le \int_t^\infty dt |\frac{\partial B_4}{\partial \theta_4^i}| \\
&\le \epsilon^3 C_1 \frac{1}{t^3}, \tag{6.46} \\
|\delta J_4^i(t) - \delta J_4^i{}_\infty| &\le \int_t^\infty dt |\frac{\partial^2 B_4}{\partial \theta_4^i \partial \theta_4} \delta\theta_4 + \frac{\partial^2 B_4}{\partial \theta_4^i \partial J_4} \delta J_4| \\
&\le \frac{\epsilon^2}{t^2} (C_2 |\delta J_0| + \epsilon C_3 |\delta\theta_0|), \tag{6.47}
\end{aligned}
$$

for some positive C_1, C_2, C_3. As $V(t)$ in the proposition, we adopt

$$V(t) = \frac{1}{t-1} \int_1^t dt \delta \frac{1}{(\mu \cdot J_4)^{1/2}}. \tag{6.48}$$

By using (6.46), (6.47), (6.48), for $t_1, t_2 \ge 2$, we obtain

$$
\begin{aligned}
|V(t_1) - V(t_2)| &= |\frac{1}{t_1 - 1} \int_{t_2}^{t_1} dt \delta \frac{1}{(\mu \cdot J_4)^{1/2}} + \frac{(t_2 - t_1)}{(t_1 - 1)(t_2 - 1)} \int_1^{t_2} dt \delta \frac{1}{(\mu \cdot J_4)^{1/2}}| \\
&\le |\frac{-1}{t_1 - 1} \int_{t_1}^{t_2} dt (-\frac{1}{2}) \frac{1}{(\mu \cdot J_4)^{3/2}} \mu \cdot \delta J_4 - \frac{-1}{t_1 - 1} \int_{t_1}^{t_2} dt (-\frac{1}{2}) \frac{1}{(\mu \cdot J_4{}_\infty)^{3/2}} \mu \cdot \delta J_4{}_\infty| \\
&+ |\frac{t_2 - t_1}{(t_1 - 1)(t_2 - 1)} \int_1^{t_2} dt (-\frac{1}{2}) \frac{1}{(\mu \cdot J_4)^{3/2}} \mu \cdot \delta J_4 \\
&- \frac{t_2 - t_1}{(t_1 - 1)(t_2 - 1)} \int_1^{t_2} dt (-\frac{1}{2}) \frac{1}{(\mu \cdot J_4{}_\infty)^{3/2}} \mu \cdot \delta J_4{}_\infty| \\
&\le |\frac{1}{t_1} - \frac{1}{t_2}| \epsilon^2 (C_4 |\delta J_0| + \epsilon C_5 |\delta\theta_0|), \tag{6.49}
\end{aligned}
$$

which yields

$$|V(t) - V(t = \infty)| \le \frac{\epsilon^2}{t} (C_4 |\delta J_0| + \epsilon C_5 |\delta\theta_0|). \tag{6.50}$$

We complete the proof.

Proof End

Using the above results to Eq.(6.5) we can see that for sufficiently large t,

$$Z \to -\frac{2}{3} (\mu \cdot J + K)^{1/2} V(t = \infty). \tag{6.51}$$

Therefore for the mode satisfying $V(t = \infty) \ne 0$ which corresponds to the adiabatic growing mode, Z stays constant, and for other $(2N - 1)$ modes satisfying $V(t = \infty) = 0$, Z decays at most in proportion to $1/t$. Then the uniqueness of the growing mode are derived.

Because of the factor as

$$\frac{1}{\sum_i \mu_i J^i (1 - \cos 2\theta^i)}, \tag{6.52}$$

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all the terms in Eq.(6.5) except (6.51) become boundlessly large when the phase flow $\theta(t)$ approaches $\theta^i = 0$ or π ($i = 1, \cdots, N$) boundlessly, although the numerator are not taken into account approaching zero. Does our system spend time near $\theta^i = 0$ or π ($i = 1, \cdots, N$) where the Bardeen parameter becomes large? The answer is as follows. Our track approaches the singular point boundlessly infinite times. The distribution of the singularities in the time axis is probabilistic and random. This problem can be regarded in the viewpoint of the ergodic theory. We formulate this as follows.

Proposition5

Under the condition

(i)' If the equality $\mu \cdot k = 0$ holds for some integral-valued vector $k = (k^1, \cdots, k^N)$, $k = 0$ holds.

and the assumptions (ii), (iii) in the above proposition,

there exists some positive ϵ_0 such that;

for an arbitrary ϵ satisfying

$$0 < \epsilon < \epsilon_0, \tag{6.53}$$

the following statement holds.

For all Riemannian integrable function $f(\theta^1, \cdots, \theta^N)$ which is 2π-periodic in $\theta = (\theta^1, \cdots, \theta^N)$, the equality

$$\lim_{T \to \infty} \frac{1}{T-1} \int_1^T dt f(\theta(t)) = \overline{f}, \tag{6.54}$$

holds where

$$\overline{f} = \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} d^N\theta f(\theta). \tag{6.55}$$

We prove this proposition in the appendix. As a collorary of the proposition, we obtain

Corollary

Under the assumtions in the above proposition, as for a definition function $\phi(\theta)$ of the Jordan measurable set $A \subset \prod_{i=1}^N [0, 2\pi]$;

$$\phi(\theta) = \begin{cases} 1 & (\theta \in A) \\ 0 & (\theta \in \prod_{i=1}^N [0, 2\pi] \setminus A) \end{cases} \tag{6.56}$$

the equality

$$\lim_{T \to \infty} \frac{1}{T-1} \int_1^T dt \phi(\theta(t)) = \frac{1}{(2\pi)^N} \tau(A) \tag{6.57}$$

holds where $\tau(A)$ is a measure of the set A.

Proof

When the set A is Jordan measurable, its definition function $\phi(\theta)$ is Riemannian integrable. Applying the above proposition to $\phi(\theta)$, we obtain

$$\begin{aligned} \lim_{T \to \infty} \frac{1}{T-1} \int_1^T dt \phi(\theta(t)) &= \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} d^N\theta \phi(\theta) \\ &= \frac{1}{(2\pi)^N} \tau(A). \end{aligned} \tag{6.58}$$

Proof End

From the above corollary, it can be seen that when the masses of the scalar fields are irrationally independent, each track distributes uniformly in the torus T^N, more accurately saying, the period for which an arbitrary track stays in an arbitrary domain A is proportional to the measure of A. Let us adopt the domain which contains the singular point $\theta^i = 0$ or π $(i = 1, \cdots, N)$ as A. Then we can evaluate how long the Bardeen parameter becomes singular. But we cannot say when the Bardeen parameter becomes singular. The occurrence of singularities is essentially probabilistic and random. We can find no rule in a sequence of times when Z becomes singular.

§7 Summary and Discussion

During the reheating, the multiple scalar fields oscillate arond the local minimum of the potential. In order to understand the evolutionary behavior of the cosmological perturbation on superhorizon scales during reheating, we investigate that in a universe dominated by multiple oscillating scalar fields. The dynamical behavior of the cosmological perturbations on superhorizon scales is well approximated in terms of that of the long wavelength limit, and the latter can be constructed from the evolution of corresponding exactly homogeneous universe. These facts provide us with a strong tool to investigate the evolution of the cosmological perturbation on superhorizon scales including the conservation of the Bardeen parameter. In order to solve the evolutions of the cosmological perturbations in the long wavelength limit, we need know the evolution of the exactly homogeneous system. Since the scalar fields oscillate rapidly, we use the action angle variables. When this problem is formulated in the hamiltonian form, as the result of the expansion of the universe and the interactions between the scalar fields the perturbed part depending on the angle variable appears. We found that it is bounded by the negative power of the time and that by choosing the canonical transformation properly, the perturbed part, in particular the power of the time, can be decreased so that the transformed hamiltonian can be solved easily. By using the long wavelength limit method, and the technique of the canonical perturbation theory, we prove the fact that if the ratio of frequncies of the scalar fields are incommensurable, for the adiabatic growing mode Bardeen parameter stays constant and for the other $(2N - 1)$ modes, the Bardeen parameter decays. In addition we show that the Bardeen parameter diverges probabilistically. From the viewpoint of the ergodic theory, we evaluate the period for which the Bardeen parameter diverges.

In this paper, since the evolution of the cosmological perturbations are assumed to be generated from the perturbations of the exactly homogeneous solutions, we can only take into account the interactions between the homogenous modes, although in the realistic reheating scenario the interactions between the non-homogeneous modes cannot be neglected and may be rather strong. When we think that the analysis of the latter are too complicated to be treated, the investigation of this paper are important, since we can investigate analytically, and does not specify the details of the interactions. Although the analysis are confined in the non-resonant case, the elimination of fast phases by the canonical transformation is thought to be still effective also in resonance case. The analysis in resonant case is in future publications.

Our final goal is understanding of the evolutionary behaviors of the cosmological per-

turbations during reheating. We have shown the constancy of the Bardeen parameter in two extreme cases, that is, in the nearly integrable case corresponding to this paper, and the chaotic case where the discription by the two component fluid model [8] becomes good. So in order to achieve our purpose completely, general numerical calculations taking into account the interactions between the non-homogeneous modes, taking its stands between the above two extreme cases are necessary.

§8 Appendix; Mathematical Preliminary

We give a little knowledge of the complex analysis and the fourier analysis, since it is used in the physical investigations of this paper. These propositions are fundamental when we evaluate the upper bound of the generating functions of the canonical transformations which decrease the size and the power of the time of the perturbed part of the hamiltonian.

The following proposition is often used when we evaluate the derivative with respect to the canonical variables, for example the defference between the original variables and the transformed variables and the coefficients appearing in the perturbations of the canonical equations of motion.

Proposition

$f(x)$ is analytic and the inequality

$$|f(x)| \leq C \tag{8.1}$$

holds for some positive C on the domain D. Then for an arbitrary positive δ, in the domain $D - \delta = \{x \in D;$ the δ neighborhood of x is contained in $D\}$, the inequalities

$$|\frac{\partial f}{\partial x}| \leq \frac{C}{\delta}, \tag{8.2}$$

$$|\frac{\partial^2 f}{\partial x^2}| \leq \frac{2C}{\delta^2}, \tag{8.3}$$

hold.

Proof

Use the Cauchy equation

$$f(x) = \frac{1}{2\pi i} \oint dy \frac{f(y)}{y - x}. \tag{8.4}$$

Proof End

The following proposition guarantee the fact that the derivative with respect to the time decreases the power of the time.

Proposition

$t^m S(t)$ is analytic function of $s = 1/t$ on the domain $s \in C(\rho)$ where

$$C(\rho) = \{s; |s| \leq 1 + \rho, |\text{Im} s| \leq \rho\} \tag{8.5}$$

and the inequality

$$|t^m S| \leq C \tag{8.6}$$

holds for some positive C. Then for an arbitrary positive δ, on the domain $C(\rho - \delta)$ $t^{m+1} \partial S / \partial t$ is analytic and the inequality

$$|t^{m+1} \frac{\partial S}{\partial t}| \leq C(m + \frac{1 + \rho}{\delta}) \tag{8.7}$$

holds.

Proof

By differentiating the equation

$$\frac{1}{s^m}S(\frac{1}{s}) = \oint dz \frac{1}{z-s}\frac{1}{z^m}S(\frac{1}{z}), \tag{8.8}$$

we obtain

$$\frac{1}{s^{m+1}}S'(\frac{1}{s}) = -m\frac{1}{s^m}S(\frac{1}{s}) - \frac{s}{2\pi i}\oint dz \frac{1}{(z-s)^2}\frac{1}{z^m}S(\frac{1}{z}), \tag{8.9}$$

from which the proposition follows.

Proof End

Proposition

$F(\theta) = \sum_k F_k \exp ik \cdot \theta$ is periodic in $\theta = (\theta_1, \cdots, \theta_N)$ and analytic and the inequality

$$|F(\theta)| \le C \tag{8.10}$$

holds for some positive C on the domain

$$|\mathrm{Im}\theta| \le \rho. \tag{8.11}$$

Then the fourier components F_k satisfy

$$|F_k| \le C \exp(-|k|\rho) \tag{8.12}$$

where $|k|$ is defined by

$$|k| = |k_1| + \cdots + |k_N|. \tag{8.13}$$

Proof

The fourier coefficients are given by

$$F_k = \frac{1}{(2\pi)^N}\int_0^{2\pi}\cdots\int_0^{2\pi} d\theta_1\cdots d\theta_N \exp(-ik\cdot\theta)F(\theta). \tag{8.14}$$

By changing the contour into $\theta_i = x_i + i\rho_i$, $(0 \le x_i \le 2\pi$, $\rho_i = -\mathrm{sgn}(k_i)\rho)$ taking into account that contributions from the both sides cancel, F_k is evaluated as

$$\begin{aligned}|F_k| &\le \prod_i \exp(-|k_i|\rho)C \\ &= C\exp(-|k|\rho).\end{aligned} \tag{8.15}$$

Proof End

The following proposition guarantees that sufficiently small generating function defines the diffeomorphic canonical transformations. We omit the proof.

Proposition

Suppose that the mapping $T; x \to y$ is defined on the domain D. When T is differentiable and $\det\partial y/\partial x \ne 0$ and $|x-y| \le \epsilon$ on the domain D, T is diffeomorphic on the domain $D - 4\epsilon$.

§9 Appendix; Ergodic Property of Conditionally Periodic Motion

In chapter 6, we pointed out that the Bardeen parameter becomes singular at a certain point in the torus T^N. In the viewpoint of the ergodic theory, we cannot say when the Bardeen parameter becomes singular, but can say how long the Bardeen parameter singular. In this appendix, we discuss the ergodic property as for the universe dominated by multiple oscillating scalar fields whose masses are incommensurable, i.e. irrationally independent. In our system, it can be shown that an arbitrary track is dense everywhere in the torus T^N. It is referred to as a conditionary periodic motion.

We consider the system proposed in Eq.(4.17). We assume that μ is a linearly independent vector on $\mathbf{Z}^N$;

$$\mu \cdot k = 0 \Rightarrow k = 0. \tag{9.1}$$

As a first step, we prove the lemma below.

Lemma

$$\lim_{T \to \infty} \frac{1}{T-1} \int_1^T dt e^{ik \cdot \theta(t)} = 0, \tag{9.2}$$

Proof

We consider the quantity as

$$\int_1 dt F(J, \theta, t). \tag{9.3}$$

If $G(J, \theta, t)$ satisfying

$$\frac{1}{\epsilon} \frac{1}{(\mu \cdot J)^{1/2}} \sum_i \mu_i \frac{\partial G}{\partial \theta^i} = F, \tag{9.4}$$

exists, the equality

$$\int_1 dt F(J, \theta, t) = [G(J, \theta, t)]_{t=1} + \int_1 dt[-\sum_i \frac{\partial \tilde{H}}{\partial J^i} \frac{\partial G}{\partial \theta^i} + \sum_i \frac{\partial \tilde{H}}{\partial \theta^i} \frac{\partial G}{\partial J^i} - \frac{\partial G}{\partial t}] \tag{9.5}$$

holds. When $F = e^{ik \cdot \theta}$, $G = \epsilon/i(\mu \cdot k) \cdot (\mu \cdot J)^{1/2} e^{ik \cdot \theta}$. Then there exists positive M, N such that

$$|\frac{1}{T-1} \int_1^T dt e^{ik \cdot \theta}| \leq \frac{\epsilon}{(\mu \cdot k)} \frac{1}{T-1} [2M + N \ln T] \to 0 \quad (T \to \infty). \tag{9.6}$$

We complete the proof.

Proof End

Lemma

For a triangular polynomial

$$P_M(\theta) = \sum_{|k| \leq M} f_k e^{ik \cdot \theta}, \tag{9.7}$$

the equality

$$\lim_{T \to \infty} \frac{1}{T-1} \int_1^T dt P_M(\theta) = \overline{P_M} \tag{9.8}$$

holds.

Proof

The both sides of Eq.(9.8) is linear. Therefore from the above lemma, Eq.(9.8) holds.

Proof End

Lemma; Weierstrass' theorem

Suppose that $f(\mathbf{x})$ is continuous on $\prod_{i=1}^{N}[a_i, b_i]$. For an arbitrary positive ϵ, there exists a polynomial $P(\mathbf{x})$ such that

$$|f(\mathbf{x}) - P(\mathbf{x})| < \epsilon \qquad \mathbf{x} \in \prod_{i=1}^{N}[a_i, b_i]. \tag{9.9}$$

Proof

We consider the integral;

$$J_n = \int_0^1 (1 - v^2)^n dv, \tag{9.10}$$

$$J_n^* = \int_\delta^1 (1 - v^2)^n dv \quad (0 < \delta < 1). \tag{9.11}$$

Since

$$J_n > \int_0^1 (1 - v)^n dv = \frac{1}{n+1}, \tag{9.12}$$

$$J_n^* < (1 - \delta^2)^n (1 - \delta) < (1 - \delta^2)^n, \tag{9.13}$$

$$\frac{J_n^*}{J_n} < (n+1)(1 - \delta^2)^n, \tag{9.14}$$

we obtain

$$\lim_{n \to \infty} \frac{J_n^*}{J_n} = 0. \tag{9.15}$$

Without loss of generality, we can put $0 < a_i < b_i < 1$. We insert α_i, β_i such that $0 < \alpha_i < a_i < b_i < \beta_i < 1$. As a polynomial, we adopt

$$P_n(\mathbf{x}) = \frac{1}{2^N (J_n)^N} \int_C f(\mathbf{u})[1 - (u_1 - x_1)^2]^n \cdots [1 - (u_N - x_N)^2]^n du_1 \cdots du_N, \tag{9.16}$$

where

$$C = \prod_{i=1}^{N}[\alpha_i, \beta_i]. \tag{9.17}$$

$P_n(\mathbf{x})$ is a polynomial of $2n$ degree in terms of x_i, $(i = 1, \cdots, N)$. We evaluate how accurately the polynomial $P_n(\mathbf{x})$ approximates the continuous function $f(\mathbf{x})$. We introduce sets A, B defined by

$$A = \prod_{i=1}^{N}[x_i - \delta, x_i + \delta] \qquad (0 < \delta < 1), \tag{9.18}$$

$$B = \prod_{i=1}^{N}[a_i, b_i]. \tag{9.19}$$

First we evaluate

$$
\begin{aligned}
I \equiv \quad & \int_A f(\mathbf{u})[1 - (u_1 - x_1)^2]^n \cdots [1 - (u_N - x_N)^2]^n du_1 \cdots du_N \\
& - 2^N (J_n)^N f(\mathbf{x}).
\end{aligned}
\tag{9.20}
$$

In terms of $\mathbf{v} = \mathbf{u} - \mathbf{x}$, A is transferred into A' defined by

$$
A' = \prod_{i=1}^N [-\delta, \delta].
\tag{9.21}
$$

Therefore we obtain

$$
\begin{aligned}
I = \quad & \int_{A'} f(\mathbf{x})[1 - (u_1 - x_1)^2]^n \cdots [1 - (u_N - x_N)^2]^n du_1 \cdots du_N \\
& + \int_{A'} (f(\mathbf{v} + \mathbf{x}) - f(\mathbf{x}))[1 - (u_1 - x_1)^2]^n \cdots [1 - (u_N - x_N)^2]^n du_1 \cdots du_N \\
& - 2^N (J_n)^N f(\mathbf{x}).
\end{aligned}
\tag{9.22}
$$

Since $f(\mathbf{x})$ is uniformly continuous on $\mathbf{x} \in B$, there exists positive δ which depends only on ϵ $(0 < \delta < 1)$ such that

$$
|\mathbf{v}| \leq \sqrt{m}\delta
\tag{9.23}
$$

and

$$
|f(\mathbf{v} + \mathbf{x}) - f(\mathbf{x})| \leq \epsilon \qquad (\mathbf{x} \in B).
\tag{9.24}
$$

Therefore we obtain

$$
|\mathrm{I}| \leq 2^N \{(J_n)^N - (J_n - J_n^*)^N\} M + \epsilon 2^N (J_n - J_n^*)^N,
\tag{9.25}
$$

where

$$
M = \sup_{\mathbf{x} \in C} |f(\mathbf{x})|.
\tag{9.26}
$$

Next we evaluate

$$
\mathrm{II} = \int_{C \backslash A} f(\mathbf{u})[1 - (u_1 - x_1)^2]^n \cdots [1 - (u_N - x_N)^2]^n du_1 \cdots du_N.
\tag{9.27}
$$

Since in terms of $\mathbf{v} = \mathbf{u} - \mathbf{x}$, C, A are transferred into C', A' defined by

$$
\begin{aligned}
C' &= \prod_{i=1}^N [\alpha_i - x_i, \beta_i - x_i], \\
A' &= \prod_{i=1}^N [-\delta, \delta],
\end{aligned}
\tag{9.28}
$$

and

$$
C' \subset D = \prod_{i=1}^N [-1, 1],
\tag{9.29}
$$

we obtain

$$
|\mathrm{II}| \leq M 2^N \{(J_n)^N - (J_n - J_n^*)^N\}.
\tag{9.30}
$$

From Eqs.(9.25), (9.30), we obtain

$$|P_n(\mathbf{x}) - f(\mathbf{x})| \le |\mathrm{I}| + |\mathrm{II}| \le 2M\{1 - (1 - \frac{J_n^*}{J_n})^N\} + \epsilon(1 - \frac{J_n^*}{J_n})^N. \tag{9.31}$$

From Eq.(9.15), the right hand side can be smaller than 2ϵ when we adopt sufficiently large n. We complete the proof.

Proof End

Lemma

Suppose that $f(\theta^1, \cdots, \theta^N)$ is continuous and 2π-periodic. For an arbitrary positive ϵ, there exists a triangular polynomial $P(\theta^1, \cdots, \theta^N)$;

$$P(\theta^1, \cdots, \theta^N) = \frac{\alpha_0}{2} + \sum_{|k| \le M} \alpha_k e^{ik \cdot \theta}, \tag{9.32}$$

such that

$$|f(\theta) - P(\theta)| < \epsilon. \tag{9.33}$$

Proof

We define $\phi(\xi, \eta)$ by

$$\phi(\xi, \eta) = \rho_1 \cdots \rho_N f(\theta^1, \cdots, \theta^N), \tag{9.34}$$

where

$$\xi_i = \rho_i \cos \theta^i, \tag{9.35}$$
$$\eta_i = \rho_i \sin \theta^i. \tag{9.36}$$

When $\rho_i^2 = \xi_i^2 + \eta_i^2 = 1$, $(i = 1, \cdots, N)$, $\phi(\xi, \eta)$ coincide with $f(\theta)$. Since $\phi(\xi, \eta)$ is continuous for ξ_i, η_i, from Weierstrass' theorem, there exists a polynomial for ξ, η $P_n(\xi, \eta)$ which approximates uniformly $\phi(\xi, \eta)$ in a rectangle containing $\xi_i^2 + \eta_i^2 = 1$ $(i = 1, \cdots, N)$. This completes the proof.

Proof End

Proposition

For an arbitrary Riemannian integrable function $f(\theta)$, the equality

$$\lim_{T \to \infty} \frac{1}{T - 1} \int_1^T dt f(\theta(t)) = \overline{f} \tag{9.37}$$

holds.

Proof

For an arbitrary positive ϵ, there exists triangular polynomials P_1, P_2 such that

$$P_1 < f < P_2, \tag{9.38}$$
$$\overline{P_2} - \overline{P_1} < \epsilon. \tag{9.39}$$

From Eq. (9.38), we obtain

$$\frac{1}{T - 1} \int_1^T P_1(\theta(t))dt < \frac{1}{T - 1} \int_1^T f(\theta(t))dt < \frac{1}{T - 1} \int_1^T P_2(\theta(t))dt. \tag{9.40}$$

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From the above inequality, by using the ergodic equality of the triangular polynomial, we obtain

$$
\begin{aligned}
\overline{P_1} &\leq \lim_{T_0 \to \infty} \inf_{T \geq T_0} \frac{1}{T-1} \int_1^T f(\theta(t))dt \\
&\leq \lim_{T_0 \to \infty} \sup_{T \geq T_0} \frac{1}{T-1} \int_1^T f(\theta(t))dt \leq \overline{P_2}
\end{aligned}
\tag{9.41}
$$

which yields

$$
\lim_{T_0 \to \infty} \sup_{T \geq T_0} \frac{1}{T-1} \int_1^T f(\theta(t))dt - \lim_{T_0 \to \infty} \inf_{T \geq T_0} \frac{1}{T-1} \int_1^T f(\theta(t))dt < \epsilon.
\tag{9.42}
$$

Since ϵ is arbitrary, and $\overline{P_1} < \overline{f} < \overline{P_2}$, we obtain

$$
\lim_{T \to \infty} \frac{1}{T-1} \int_1^T f(\theta(t))dt = \overline{f}.
\tag{9.43}
$$

We complete the proof.
   Proof End

Acknowledgments

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